On Cobweb posets tiling problem

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SUMMARY

Kwaśniewski's cobweb posets uniquely represented by directed acyclic graphs are such a generalization of the Fibonacci tree that allows joint combinatorial interpretation for all of them under admissibility condition. This interpretation was derived in the source papers and it entailes natural enquieres already formulated therein. In our note we response to one of those problems. This is a tiling problem. Our observations on tiling problem include proofs of tiling's existence for some cobweb-admissible sequences. We show also that not all cobwebs admit tiling as defined below.

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1 Introduction

The source papers are [1, 2] from which indispensable definitions and notation are taken for granted as for example (Kwaśniewski upside - down notation $n_F \equiv F_n$ being used for mnemonic reasons [1, 2, 3]) : $F-nomial$ coefficient:

$$\binom{n}{k}_F = \frac{n_F \cdot (n-1)_F \cdot \ldots \cdot (n-k+1_F}{1_F \cdot 2_F \cdot \ldots \cdot k_F} = \frac{n_F^{\underline{k}}}{k_F!}; \quad n_F \equiv F_n$$

Nevertheless let us at first recall that cobweb poset in its original form [1, 2] is defined as a partially ordered graded infinite poset $\Pi = \langle P, \leq \rangle$, designated uniquely by any sequence of nonnegative integers $F = \{n_F\}_{n \geq 0}$ and it is represented as a directed acyclic graph (DAG) in the graphical display of its Hasse diagram. P in $\langle P, \leq \rangle$ stays for set of vertices while $\leq$ denotes partially ordered relation. See Figure 1. and note (quotation from [2, 1]):

> One refers to Φ_s as to the set of vertices at the s-th level. The population of the k-th level (”*generation*”) counts k_F different member vertices for $k > 0$ and one for $k = 0$. Here down (Fig. 1) a disposal of vertices on Φ_k levels is visualized for the case of Fibonacci sequence. $F_0 = 0$ corresponds to the empty root $\{\emptyset\}$.

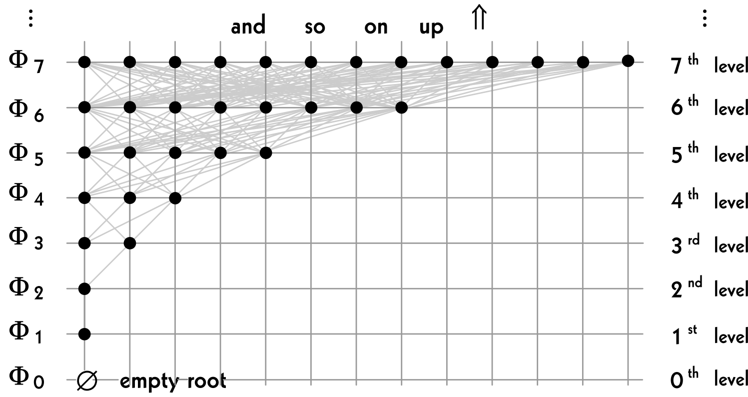

Figure 1: The s-th level in $N \times N_0$

In Kwaśniewski's cobweb posets' tiling problem one considers finite cobweb sub-posets for which we have finite number of layers $\langle \Phi_k \to \Phi_n \rangle$, where $k \leq n$, $k, n \in \mathbb{N} \cup \{0\}$ with exactly k_j vertices on Φ_j level $k \leq j \leq n$. For $k = 0$ the sub-posets $\langle \Phi_0 \to \Phi_n \rangle$ are named *prime cobweb posets* and these are those to be used - up to permutation of levels equivalence - as a block to partition finite cobweb sub-poset.

For the sake of combinatorial interpretation [1, 2] a natural numbers valued sequence F which determines a cobweb poset has to be the so-called *cobweb-admissible.*

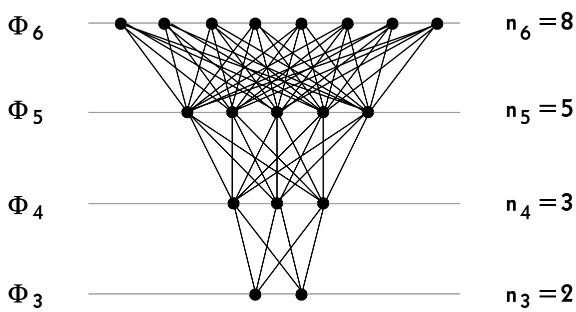

Figure 2: Display of four levels of Fibonacci numbers' finite Cobweb sub-poset

Definition 1 [2] *A natural numbers' valued sequence* $F = \{n_F\}_{n \geq 0}$, $F_0 = 1$ *is called cobweb-admissible iff*

$$\binom{n}{k}_F \in N_0 \quad for \quad k, n \in N_0.$$

$F_0 = 0$ being acceptable as $0_F! \equiv F_0! = 1$. We adopt then the convention to call the root $\{\emptyset\}$ the "empty root".

One of the problems posed in [1, 2] is the one which is the subject of our note.

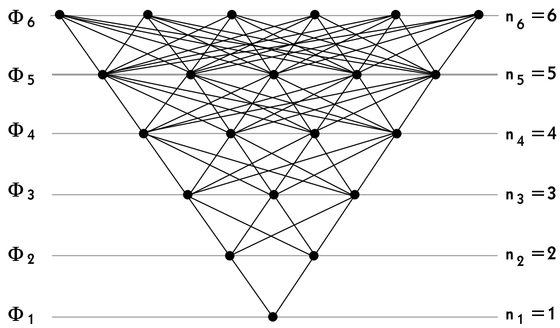

Figure 3: Display of Natural numbers' finite prime Cobweb poset

The tiling problem

Suppose now that F is a cobweb admissible sequence. Under which conditions any layer $\langle \Phi_n \to \Phi_k \rangle$ may be partitioned with help of max-disjoint blocks of established type σP_m? Find effective characterizations and/or find an algorithm to produce these partitions.

The above Kwaśniewski tiling problem [1, 2] is first of all the problem of existence of a partition an layer $\langle \Phi_k \to \Phi_n \rangle$ with max-disjoint blocks of the form σP_m defined as follows:

$$\sigma P_m = C_m[F, \sigma \langle F_1, F_2, \ldots, F_m \rangle]$$

It means that partition may contain only primary cobweb sub-posets or those obtained from primary cobweb poset P_m via permuting its levels as illustrated below (Fig. 4).

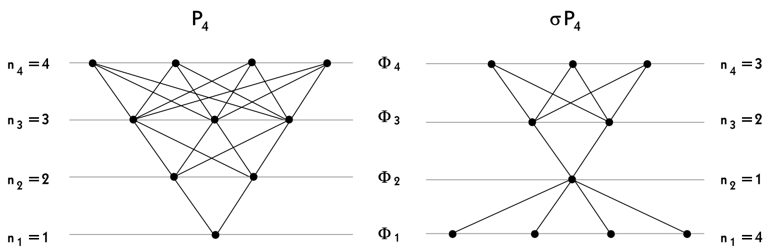

Figure 4: Display of block σP_m obtained from P_m and permutation σ

2 Example of a cobweb poset recurrent tiling algorithm - 1 (cprta1)

Now we present an algorithm to create partition of any layer $\langle \Phi_k \to \Phi_n \rangle$, $k \leq n$, $k, n \in \mathbb{N} \cup \{0\}$ of finite cobweb sub-poset specified by such F-sequences as Natural numbers and Fibonacci numbers. We shall use the abbreviation: (cprta1) algorithm. In the following Theorem 1 and Theorem 2 are existence theorems.

Theorem 1 (Natural numbers) *Consider any layer $\langle\Phi_{k+1} \to \Phi_n\rangle$ with m levels where $m = n - k$, $k \leq n$ and $k, n \in \mathbb{N} \cup \{0\}$ in a finite cobweb sub-poset, defined by the sequence of* ***natural numbers*** *i.e. $F \equiv \{n_F\}_{n\geq 0}$, $n_F = n$, $n \in \mathbb{N} \cup \{0\}$. Then there exists at least one way to partition this layer with help of max-disjoint blocks of the form σP_m.*

Max-disjoint means that the two blocks have no maximal chain in common [1, 2].
Before proving let us notice that for any $m, k \in \mathbb{N}$ such that $m + k = n$:

$$n_F = m_F + k_F \tag{1}$$

where $1_F = 1$.

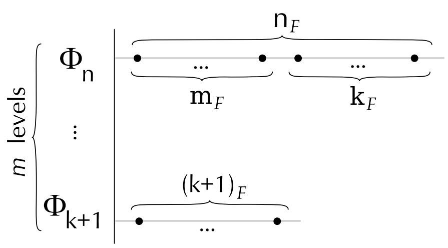

Figure 5: Picture of m levels of Cobweb poset' Hasse diagram

PROOF (cprta1) algorithm

Steep 1. There are $n_F = m_F + k_F$ vertices on the Φ_n level. Let us separate them cutting into two disjoint subsets as illustrated by the Fig.5 and cope at first with m_F vertices (Steep 2). Then we shall cope with those k_F vertices left (Steep 3).

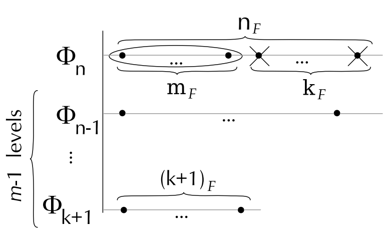

Figure 6: Picture of Steep 2

Steep 2. Temporarily we have m_F fixed vertices on Φ_n level to consider. Let us cover them by m-th level of block P_m, which has exactly m_F vertices-leafs. What was left is the layer $\langle\Phi_{k+1} \to \Phi_{n-1}\rangle$ and we might eventually partition it with smaller max-disjoint blocks σP_{m-1}, but we need not to do that. See the next step.

Steep 3. Consider now the second complementary situation, where we have k_F vertices on Φ_n level being fixed. Observe that if we *move* this level lower than

Φ_{k+1} level, we obtain exactly $\langle\Phi_k \rightarrow \Phi_{n-1}\rangle$ layer to be partitioned with max-disjoint blocks of the form σP_m. This "*move*" operation is just permutation of levels' order.

The layer $\langle\Phi_{k+1}\rightarrow\Phi_n\rangle$ may be partitioned with σP_m blocks if $\langle\Phi_{k+1}\rightarrow\Phi_{n-1}\rangle$ may be partitioned with σP_{m-1} blocks and $\langle\Phi_k\rightarrow\Phi_{n-1}\rangle$ by σP_m again. Continuing these steeps by induction, we are left to prove that $\langle\Phi_k \rightarrow \Phi_k\rangle$ may be partitioned by σP_0 blocks and $\langle\Phi_1\rightarrow\Phi_m\rangle$ by σP_m blocks which is obvious ■

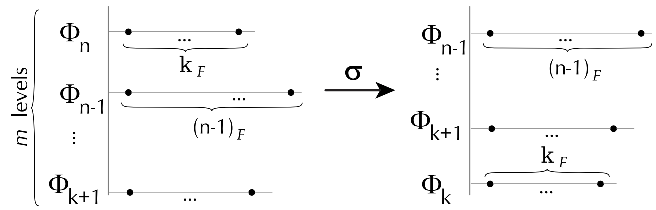

Figure 7: Picture of Steep 3

Observation 1
We know from [1, 2] (Observation 3 there) that the number of max-disjoint equip-copies of σP_m, rooted at the same fixed vertex of k-th level and ending at the n-th level is equal to

$$\binom{n}{k}_F = \binom{n}{m}_F$$

If we cut-separate family of leafs of the layer $\langle\Phi_{k+1}\rightarrow\Phi_n\rangle$, as in the proof of the Theorem 1 then the number of max-disjoint equip copies of P_{m-1} from the Steep 2 is equal to

$$\binom{n-1}{k}_F$$

However the number of max-disjoint equip copies of P_m from the Steep 3 is equal to

$$\binom{n-1}{k-1}_F$$

It gives us well-known formula of Newton's symbol recurrence:

$$\binom{n}{k}_F = \binom{n-1}{k}_F + \binom{n-1}{k-1}_F$$

in accordance with what was expected for the case $F = \mathbb{N}$ thus illustrating the combinatorial interpretation from [1, 2] in this particular case.

In the next we adapt Knuth notation for "F-Stirling numbers" of the second kind $\left\{ n \atop k \right\}_F$ as in [2] and also in conformity with Kwaśniewski notation for F-nomial coefficients [4, 1, 3]. The number of those partitions which are obtained via (cprta1) algorithm shall be denoted by the symbol $\left\{ n \atop k \right\}_F^1$.

Observation 2

Let F be a sequence matching (1). Then the number $\left\{ {n \atop k} \right\}_F^1$ of different partitions of the layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ where $n, k \in \mathbb{N},\ n, k \geq 1$ is equal to:

$$\left\{ {n \atop k} \right\}_F^1 = \binom{n_F}{m_F} \cdot \left\{ {n-1 \atop k} \right\}_F^1 \cdot \left\{ {n-1 \atop k-1} \right\}_F^1 \qquad (S_N)$$

where $\left\{ {n \atop n} \right\}_F^1 = \left\{ {n \atop n} \right\}_F = 1$, $\left\{ {n \atop 1} \right\}_F^1 = \left\{ {n \atop 1} \right\}_F = 1$, $m = n - k + 1$.

Proof

According to the Steep 1 of the proof of Theorem 1 we may choose on Φ_n level m_F vertices out of n_F ones in $\binom{n_F}{m_F}$ ways. Next recurrent steps of the proof of Theorem 1 result in formula (S_N) via product rule of counting. ■

Note. $\left\{ {n \atop k} \right\}_F^1$ is not the number of all different partitions of the layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ i.e. $\left\{ {n \atop k} \right\}_F \geq \left\{ {n \atop k} \right\}_F^1$ as computer experiments [6] show. There are much more other tilings with blocks σP_m.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 12 & & 18 & & 1 & & \\
 & 1 & & 60 & & 2160 & & 180 & & 1 & \\
1 & & 360 & & 1944000 & & 7776000 & & 2700 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}$$

Figure 8: Natural numbers' Cobweb poset tiling triangle of $\left\{ {n \atop k} \right\}_F^1$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 15 & & 1 & & & \\
 & & 1 & & \frac{24!}{4!\cdot 6!^4} & & 10395 & & 1 & & \\
 & 1 & & \frac{120!}{5!\cdot 24!^5} & & \frac{60!}{10!\cdot 6!^{10}} & & \frac{20!}{10!\cdot 2!^{10}} & & 1 & \\
1 & & \frac{720!}{6!\cdot 120!^6} & & \frac{360!}{15!\cdot 24!^{15}} & & \frac{120!}{20!\cdot 6!^{20}} & & \frac{30!}{15!\cdot 2!^{15}} & & 1 \\
 & & & & & \cdots & & & & &
\end{array}$$

Figure 9: Kwaśniewski Natural numbers' cobweb poset tiling triangle of $\left\{ {\eta \atop \kappa} \right\}_\lambda$

This is to be compared with Kwaśniewski cobweb triangle [2] (Fig. 9) for the infinite triangle matrix elements

$$\left\{ \begin{matrix} \eta \\ \kappa \end{matrix} \right\}_\lambda = \delta_{\eta,\kappa\lambda} \frac{\eta!}{\kappa!\lambda!^\kappa}$$

counting the number of partitions with block sizes all equal to λ.
Here $const = \lambda = m_F!, m = n - k + 1$ and

$$\eta = n_F^{\underline{m}}, \quad \kappa = \binom{n}{k-1}_F$$

The inequality $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F^1 \leq \left\{ \begin{smallmatrix} \eta \\ \kappa \end{smallmatrix} \right\}_\lambda$ gives us the rough upper bound for the number of tilings with blocks of established type σP_m.

Theorem 2 (Fibonacci numbers) *Consider any layer $\langle \Phi_{k+1} \to \Phi_n \rangle$ with m levels where $m = n - k$, $k \leq n$ and $k, n \in \mathbb{N} \cup \{0\}$ in a finite cobweb sub-poset, defined by the sequence of Fibonacci numbers i.e. $F \equiv \{n_F\}_{n \geq 0}, n_F \in \mathbb{N} \cup \{0\}$. Then there exists at least one way to partition this layer with help of max-disjoint blocks of the form σP_m.*

The proof of the Theorem 2 for the Fibonacci sequence F is similar to the proof of Theorem 1. We only need to notice that for any $m, k \in \mathbb{N}$, $m > 1$, $m + k = n$ the following identity takes place:

$$n_F = (m + k)_F = (k + 1)_F \cdot m_F + (m - 1)_F \cdot k_F \tag{2}$$

where $1_F = 2_F = 1$.

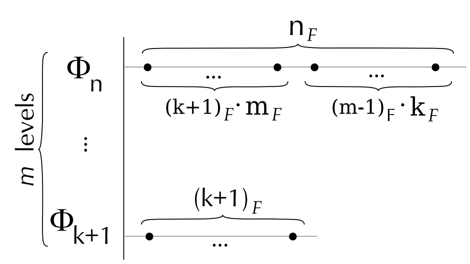

Figure 10: Picture of m levels' layer of Fibonacci Cobweb graph

Proof

The number of leafs on the Φ_n layer is the sum of two summands $\kappa \cdot m_F$ and $\mu \cdot k_F$, where $\kappa = (k + 1)_F$, $\mu = (m - 1)_F$, (Fig. 10) therefore as in the proof of the Theorem 1 we consider two parts. At first we have to partition κ layers $\langle \Phi_{k+1} \to \Phi_{n-1} \rangle$ with blocks σP_{m-1} and μ layers $\langle \Phi_k \to \Phi_{n-1} \rangle$ with σP_m. The rest of the proof goes similar as in the case of the Theorem 1 ■

Theorem 2 is a generalization of Theorem 1 corresponding to *const* $= \kappa, \mu = 1$ case.

Observation 3
The number of max-disjoint equip copies of P_{m-1} which partition κ layers $\langle\Phi_{k+1} \to \Phi_{n-1}\rangle$ is equal to

$$\kappa\binom{n-1}{k}_F = (k+1)_F\binom{n-1}{k}_F$$

However this number of max-disjoint equip copies of P_m which partition μ layers $\langle\Phi_k \to \Phi_{n-1}\rangle$ is equal to

$$\mu\binom{n-1}{k-1}_F = (m-1)_F\binom{n-1}{k-1}_F$$

Therefore the sum corresponding to the Step 2 and to the Step 3 is the well known recurrence relation for Fibonomial coefficients [5, 1, 2, 3]

$$\binom{n}{k}_F = (k+1)_F\binom{n-1}{k}_F + (m-1)_F\binom{n-1}{k-1}_F$$

in accordance with what was expected for the case F being now Fibonacci sequence thus illustrating the combinatorial interpretation from [1, 2] in this particular case.

Observation 4
Let F be a sequence matching (2). Then the number $\left\{ {n \atop k} \right\}^1_F$ of different partitions of the layer $\langle\Phi_k \to \Phi_n\rangle$ where $n,k \in \mathbb{N}, n,k \geq 1$ is equal to:

$$\left\{ {n \atop k} \right\}^1_F = \frac{F_n!}{(F_m!)^\kappa \cdot (F_{k-1}!)^\mu} \cdot \left\{ {n-1 \atop k} \right\}^1_F \cdot \left\{ {n-1 \atop k-1} \right\}^1_F \qquad (S_F)$$

where $\left\{ {n \atop n} \right\}^1_F = \left\{ {n \atop n} \right\}_F = 1$, $\left\{ {n \atop n-1} \right\}^1_F = \left\{ {n \atop n-1} \right\}_F = 1$, $\left\{ {n \atop 1} \right\}^1_F = \left\{ {n \atop 1} \right\}_F = 1$, $\kappa = k_F, \mu = (m-1)_F, m = n-k+1$, $F_n! = 1 \cdot 2 \cdot \ldots \cdot (n_F - 1) \cdot n_F$.

Proof

According to the Steep 1 of the proof of Theorem 2 we may choose on n-th level m_F vertices κ times and next $(k-1)_F$ vertices μ times out of n_F ones in $\frac{F_n!}{(F_m!)^\kappa \cdot (F_{k-1}!)^\mu}$ ways. Next recurrent steps of the proof of Theorem 2 result in formula (S_F) via product rule of counting ■

Observation 4 becomes Observation 2 once we put $const = \kappa, \mu = 1$.

Easy example
For cobweb-admissible sequences F such that $1_F = 2_F = 1$, $\left\{ {n \atop n-1} \right\}^1_F = \left\{ {n \atop n-1} \right\}_F = 1$ as obviously we deal with the perfect matching of the bipartite graph which is very exceptional case (Fig. 11).

Note. As in the case of Natural numbers for F-Fibonacci numbers $\left\{ {n \atop 1} \right\}^1_F$ is not the number of all different partitions of the layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ i.e. $\left\{ {n \atop k} \right\}_F \geq \left\{ {n \atop k} \right\}^1_F$ as computer experiments [6] show. There are much more other tilings with blocks σP_m.

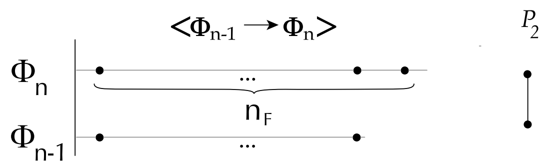

Figure 11: Easy example picture

This is to be compared with Kwaśniewski cobweb triangle [2] for the infinite triangle matrix elements (Fig. 13)

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 3 \quad 1 \quad 1 \\
1 \quad 60 \quad 90 \quad 1 \quad 1 \\
1 \quad 20160 \quad 6048000 \quad 226800 \quad 1 \quad 1 \\
\cdots
\end{array}$$

Figure 12: Fibonacci numbers' cobweb poset tiling triangle of $\left\{ {n \atop k} \right\}^1_F$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 15 \quad 1 \quad 1 \\
1 \quad \frac{30!}{5! \cdot 6!^5} \quad \frac{30!}{15! \cdot 2!^{15}} \quad 1 \quad 1 \\
1 \quad \frac{240!}{8! \cdot 30!^8} \quad \frac{240!}{40! \cdot 6!^{40}} \quad \frac{120!}{60! \cdot 2!^{60}} \quad 1 \quad 1 \\
\cdots
\end{array}$$

Figure 13: Kwaśniewski Fibonacci numbers' cobweb tiling triangle of $\left\{ {\eta \atop \kappa} \right\}_\lambda$

3 Other tiling sequences

Definition 2 *The cobweb admissible sequences that designate cobweb posets with tiling are called cobweb tiling sequences.*

3.1 Easy examples

The above method applied to prove tiling existence for Natural and Fibonacci numbers relies on the assumptions (1) or (2). Obviously these are not the only sequences that do satisfy recurrences (1) or (2). There exist also other cobweb tiling sequences beyond the above ones with different initial values.

There exist also cobweb admissible sequences determining cobweb poset with *no* tiling of the type considered in this note.

Example 1 $n_F = (m+k)_F = m_F + k_F$, $n \geq 1$ (0_F = corresponds to one "*empty root*" $\{\emptyset\}$ - compare with Definition 1)

This might be considered a sample example illustrating the method. For example if we choose $1_F = c \in \mathbb{N}$, we obtain the class of sequences $n_F = c \cdot n$ for $n \geq 1$. Naturally layers of such cobweb posets designated by the sequence satisfying (1) for $n \geq 1$ may also be partitioned according to (cprta1).

Example 1.5 $1_F = 1, n_F = c \cdot n, n > 1$ (0_F = corresponds to one "*empty root*" $\{\emptyset\}$) This might be considered another sample example now illustrating the "*shifted*" method named (cpta2). For example if we choose $2_F = c \in \mathbb{N}$, while $1_F = 1$, we obtain the class of sequences $1_F = 1$ and $n_F = c \cdot n$ for $n > 1$. Layers of such cobweb posets designated by these sequences may also be partitioned.

Observation 5 Algorithm (cpta2)
Given any (including cobweb-admissible) sequence $A \equiv \{n_A\}_{n\geq 0}$, $s \in \mathbb{N} \cup \{0\}$ *let us define* shift *unary operation* $\oplus_s$ *as follows:*

$$\oplus_s A = B, \qquad n_B = \begin{cases} 1 & n < s \\ (n-s)_A & n \geq s \end{cases}$$

where $B \equiv \{n_B\}_{n\geq 0}$. *Naturally* $\oplus_0$ = *identity. Then the following is true. If a sequence A is cobweb-tiling sequence then B is also cobweb-tiling sequence.*

For example this is the case for $A = 1, 2, 3, 4, \ldots$, $\oplus_3 A = 1, 1, 1, 1, 2, 3, 4, \ldots$.

Example 2 $n_F = m_F \cdot k_F$

If we choose $1_F = c \in \mathbb{N}$, we obtain the class of sequences $n_F = c^n, n \geq 0$. We can also consider more general case $n_F = \alpha \cdot m_F \cdot k_F$, where $\alpha \in \mathbb{N}$ which gives us the next class of tiling sequences $n_F = \alpha^{n-1} \cdot c^n, n \geq 1, 0_F = 1$ and layers of such cobweb posets can be partitioned by (cprta1) algorithm. For example: $1_F = 1, \alpha = 2 \to F = 1, 1, 2, 4, 8, 16, 32, \ldots$ or $1_F = 2 \to F = 1, 2, 4\alpha, 8\alpha^2, 16\alpha^3, \ldots$

Example 3 $n_F = (m+k)_F = (k+1)_F \cdot m_F + (m-1)_F \cdot k_F$

Here also we have infinite number of cobweb tiling sequences depending on the initial values chosen for the recurrence $(k+2)_F = 2_F(k+1)_F + k_F, k \geq 0$. For example: $1_F = 1$ and $2_F = 2 \rightarrow F = 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$ Note that this is not shifted Fibonacci sequence as we use recurrence (2) which depends on initial conditions adopted. Next $1_F = 1$ and $2_F = 3 \rightarrow F = 1, 3, 10, 33, 109, 360, 1189, \ldots$ Note that this is not remarcable Lucas sequence [7] (reference [7] was indicated to me by A.K.Kwaśniewski).

Neither of sequences : shifted Fibonacci nor Lucas sequence satisfy (2) neither these are cobweb admissible sequences as is the case of Catalan, Motzkin, Bell or Euler numbers.

The proof of tiling existence leads to many easy known formulas for sequences, where we use multiplications of terms m_F and/or k_F, like $n_F = \alpha \cdot k_F$, $n_F = \alpha \cdot m_F k_F$, $n_F = \alpha \cdot (m \pm \beta)_F k_F$, where $\alpha, \beta \in \mathbb{N}$, $n = m + k$ and so on.

This are due to the fact that in the course of partition's existence proving with (cprta1) partition of layer $\langle \Phi_{k+1} \rightarrow \Phi_n \rangle$ existence relies on partition's existence of smaller layers $\langle \Phi_{k+1} \rightarrow \Phi_{n-1} \rangle$ and/or $\langle \Phi_k \rightarrow \Phi_{n-1} \rangle$.

In what follows we shall use an at the point product of two cobweb-admissible sequences giving as a result a new cobweb admissible sequence - cobweb tiling sequences included to which the above described treatment (cprta1) applies.

3.2 Beginnings of the cobweb-admissible sequences production

Definition 3 *Given any two cobweb-admissible sequences $A \equiv \{n_A\}_{n \geq 0}$ and $B \equiv \{n_B\}_{n \geq 0}$, their at the point product C is given by*

$$A \cdot B = C \qquad C \equiv \{n_C\}_{n \geq 0}, \; n_C = n_A \cdot n_B$$

It is obvious that $A \cdot B = C$ is also cobweb admissible and

$$\binom{n}{k}_{A \cdot B} = \frac{n_A^{\underline{k}}}{k_A!} \cdot \frac{n_B^{\underline{k}}}{k_B!} = \binom{n}{k}_A \cdot \binom{n}{k}_B \in \mathbb{N} \cup \{0\}$$

Example 4 *Almost constant sequences C_t*

$$C_t = \{n_C\}_{n \geq 0} \qquad \text{where } const = n_C = t \in \mathbb{N} \text{ for } n > 0, 0_F = 1.$$

as for example $C_5 = 1, 5, 5, 5, 5, \ldots$ are trivially cobweb-admissible and cobweb tiling sequences - see next example.

In the following I denotes unit sequence $I \equiv \{1\}_{n \geq 0}$; $I \cdot A = A$.

Example 5 *Not diminishing sequence $A_{c,M}$*

If we multiply i-th term (where $i \geq M \geq 1, M \in \mathbb{N}$) of sequence I by any constant $c \in \mathbb{N}$, then the product cobweb admissible sequence is $A_{c,M}$.

$$A_{c,M} \equiv \{n_A\}_{n \geq 0} \qquad \text{where } n_A = \begin{cases} 1 & 1 \leq n < M \\ c & n \geq M \end{cases}$$

as for example $A_{5,10} = 1, \underbrace{1, \ldots, 1}_{10}, 5, 5, 5, \ldots$ or more general example $A_{3,2,10} = 1, \underbrace{3, \ldots, 3}_{10}, 6, 6, 6, \ldots$ Clearly sequences of this type are cobweb admissible and cobweb tiling sequences.

Indeed. Each of level of layer $\langle \Phi_k \to \Phi_n \rangle$ has the same or more vertices than each of levels of the block σP_m. If not the same then the number of vertices from the block σP_m divides the number of vertices at corresponding layer's level. This is how (cprta2) applies.

Note. The sequence $A_{3,2,10}$ is a product of two sequences from Example 4, $A = 1, 3, 3, 3, 3, 3, 3, \ldots$ and $B' = \oplus_{10} B = 1, \ldots, 1, 2, 2, 2, \ldots$ where $B = 1, 2, 2, 2, 2, 2, 2, \ldots$, then $A \cdot B' = A_{3,2,10} = 1, \underbrace{3, \ldots, 3}_{10}, 6, 6, 6, \ldots$

Example 6 *Periodic sequence* $B_{c,M}$

A more general example is supplied by

$$B_{c,M} \equiv \{n_B\}_{n \geq 0} \qquad \text{where } n_B = \begin{cases} 1 & M \nmid n \vee n = 0 \\ c & M | n \end{cases}$$

where $c, M \in \mathbb{N}$. Sequences of above form are cobweb tiling, as for example $B_{2,3} = \underbrace{1, 1, 2}_{3}, 1, 1, 2, \ldots$, $B_{7,4} = \underbrace{1, 1, 1, 7}_{4}, 1, 1, 1, 7, \ldots$ Indeed.

Proof Consider any layer $\langle \Phi_k \to \Phi_n \rangle$, $k \leq n$, $k, n \in \mathbb{N} \cup \{0\}$, with m levels:

For $m < M$, the block P_m has one vertex on each of levels. The tiling is trivial. For $m \geq K$, the sequence $B_{c,M}$ has a period equal to M, therefore any layer of m levels has the same or larger number of levels with c vertices than the block σP_m, if layer's level has more vertices than corresponding level of block σP_m then the quotient of this numbers is a natural number i.e. $1|c$, thus the layer can be partitioned by one block P_m or by c blocks σP_m ■

Observation 6
The at the point product of the above sequences gives us occasionally a method to produce Natural numbers as well as expectedly other cobweb-admissible sequences with help of the following algorithm.

Algorithm for natural numbers' generation (cta3)

$N(s)$ denotes a sequence which first s members is next Natural numbers i.e. $N(s) \equiv \{n_N\}_{n \geq 0}$, where $n_N = n$, for $n = 1, 2, \ldots, s$, p, p_n - prime numbers.

1. $N(1) = \mathbf{I} = 1, 1, 1, \ldots$
2. $N(2) = N(1) \cdot B_{2,2} = 1, 2, 1, 2, 1, 2, \ldots$
3. $N(3) = N(2) \cdot B_{3,3} = 1, 2, 3, 2, 1, 6, \ldots$

n. $N(n) = N(n-1) \cdot \mathbf{X}$

Consider n:

1. let n be prime, then $\neg\exists_{1 \neq i \in [n-1]} i | n \Rightarrow n_N = 1 \Rightarrow \mathbf{X} = B_{n,n}$
2. let $n = p^m$, $1 < m \in \mathbb{N}$, then $n_N = p^{m-1} \Rightarrow \mathbf{X} = B_{p,n}$
3. let $n = \prod_{s=1}^{u} p_s^{m_s}$, where $p_i \neq p_j$ for $i \neq j$, $m_i \geq 1$, $i = 1, 2, \dots, u$, $u > 1$

   $\forall_{i \in [u]} p_i^{m_i} < n \Rightarrow n_N = \mathrm{LCD}\left(\{p_i^{m_i} : i = 1, 2, \dots, u\}\right)$
   $\wedge \forall_{i \neq j} \mathrm{GCD}(p_i^{m_i}, p_j^{m_j}) = 1 \Rightarrow n_N = \prod_{s=1}^{u} p_s^{m_s} \Rightarrow \mathbf{X} = \mathbf{I}$

where lowest common denominator or least common denominator (LCD) and greatest common divisor (GCD) abbreviations were used.

Concluding

$$N(n) = N(n-1) \cdot B_{h_n,n} \overset{n \to \infty}{\longrightarrow} \mathbb{N}$$

$$h_n = \begin{cases} p & n = p^m, \quad \mathbb{N} \ni m \geq 1 \\ 1 & n = \prod_{s=1}^{u>1} p_s^{m_s}, \quad \mathbb{N} \ni m_s \geq 1 \end{cases}$$

while $\{h_n\}_{n \geq 1} = 1, 2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, \dots$

M	$h_i \backslash n_F$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
1	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	**2**	1	**2**	1	**2**	1	**2**	1	**2**	1	**2**	1	**2**	1	**2**	
3	**3**	1	1	**3**	1	1	**3**	1	1	**3**	1	1	**3**	1	1	
4	**2**	1	1	1	**2**	1	1	1	**2**	1	1	1	**2**	1	1	...
5	**5**	1	1	1	1	**5**	1	1	1	1	**5**	1	1	1	1	
6	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
7	**7**	1	1	1	1	1	1	**7**	1	1	1	1	1	1	**7**	
8	**2**	1	1	1	1	1	1	1	**2**	1	1	1	1	1	1	
	F=	1	2	3	4	5	6	7	8	3	10	1	12	1	14	...

Figure 14: Display of eight steeps of algorithm (cta3)

As for the Fibonacci sequence we expect the same statement to be true for $n \to \infty$ bearing in mind those properties of Fibonacci numbers which make them an effective tool in Zeckendorf representation of natural numbers. For the Fibonacci numbers the would be sequence $\{h_n\}_{n \geq 1}$ is given by $\{h_n\}_{n \geq 1} = 1, 1, 2, 3, 5, 4, 13, 7, 17, 11, 89, 6, \dots$

We end up with general observation - rather obvious but important to be noted.

Theorem 3 *Not all cobweb-admissible sequences are cobweb tiling sequences.*

Proof

It is enough to give an appropriate example. Consider then a cobweb-admissible sequence $F = A \cdot B = 1, 2, 3, 2, 1, 6, 1, 2, 3, \ldots$, where $A = 1, 2, 1, 2, 1, 2 \ldots$ and $B = 1, 1, 3, 1, 1, 3, \ldots$ are both cobweb admissible and cobweb tiling. Then the layer $\langle \Phi_5 \rightarrow \Phi_7 \rangle$ can not be partitioned with blocks σP_3 as the level Φ_5 has one vertex, level Φ_5 has six while Φ_5 has one vertex again (Fig 15).

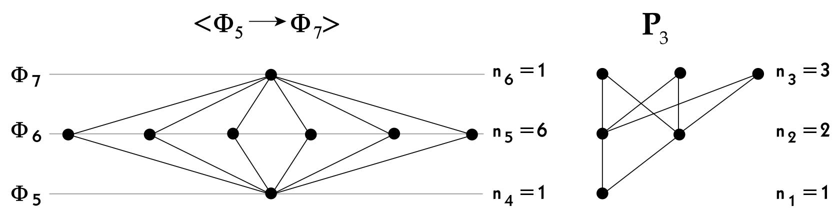

Figure 15: Picture proof of Theorem 3

Corollary *The at the point product of two tiling sequences does not need to be a tiling sequence.*

However for $A = 1, 2, 1, 2, \ldots$ and $B = 1, 1, 3, 1, 1, 3, \ldots$ cobweb tiling sequences their product $F = A \cdot B = 1, 2, 3, 2, 1, 6, 1, \ldots$ is not a cobweb tiling sequence.

A natural question - enquire is anyhow still ahead [1, 2]. Find the effective characterizations and or algorithms for a cobweb admissible sequence to be a cobweb tiling sequence.

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